Generalized spectral resolution. Let $\{|a_i)\}$ refer to an arbitrary basis in the real inner-product space $\mathcal{V}_n$. Since neither orthogonality nor normality are assumed, we have

$$(a_i|a_j) = g_{ij}$$

where $\|g_{ij}\|$ is real, symmetric and (by linear independence) non-singular. The generic element $|x)\in\mathcal{V}_n$ can be developed

$$|x) = |a_k)\,x^k$$

which gives

$$(a_j|x) = g_{jk}x^k$$

Writing $\|g_{ij}\|^{-1} = \|g^{ij}\|$ we have

$$g^{ij}(a_j|x) = g^{ij}g_{jk}x^k = \delta^i{}_k x^k = x^i$$

giving

$$|x) = |a_i)g^{ij}(a_j|x) \quad : \quad \text{all } |x)$$

from which we conclude that

$$|a_i)g^{ij}(a_j| = \mathbb{I}$$

Introduce now into $\mathcal{V}_n$ a second non-orthogonal basis with elements

$$|A^j) = |a_i)g^{ij} \quad \text{similarly} \quad (A^i| = g^{ij}(a_j|$$

which supply these alternative constructions of the unit matrix:

$$|A^i)(a_i| = |A^i)g_{ij}(A^j| = |a_j)(A^j| = \mathbb{I}$$

Moreover

$$(A^i|a_j) = g^{ik}(a_k|a_j) = g^{ik}g_{kj} = \delta^i{}_j$$

which is to say:

$$|A^i) \perp \text{all } |a_j) : j \neq i$$

The non-orthonormal bases $\{|a_i)\}$ and $\{|A^j)\}$ are said to be "biorthogonal" (or "reciprocal").[1]

Look now to the matrices

$$\mathbb{P}_i = |a_i)(A^i| \quad \text{and} \quad \mathbb{Q}_i = |A^i)(a_i| \quad \text{no summation on } i$$

where the index placement on $\mathbb{P}_i$ and $\mathbb{Q}_i$ is merely conventional (intended to convey no mathematical meaning). Those (I look only to $\mathbb{P}_i$; similar remarks pertain to $\mathbb{Q}_i$) seen to be orthogonal projection matrices

$$\mathbb{P}_i\mathbb{P}_j = |a_i)(A^i|a_j)(A^j| = |a_i)\delta^i{}_j(A^j| = \begin{cases} \mathbb{P}_i & : \quad i = j \\ \mathbb{O} & : \quad i \neq j \end{cases}$$

and have already been seen to be complete: $\sum_i \mathbb{P}_i = \mathcal{I}$. They project onto

[1] When $\{|a_i)\}$ is in fact orthonormal ($g^{ij} = \delta^{ij}$) the distinction between $\{|a_i)\}$ and $\{|A^i)\}$—as also between $\{(a_i|\}$ and $\{(A^i|\}$—evaporates.

1-spaces (rays); specifically

$$\left.\begin{aligned} \text{right action:}\quad & \mathbb{P}_i|x) = |a_i)\,x^i \\ \text{left action:}\quad & (x|\mathbb{P}_i = x_i\,(A^i| \end{aligned}\right\} \quad : \quad \text{no summation on } i$$

Given an arbitrary square matrix $\mathbb{M}$ we have

$$\begin{aligned} \mathbb{M} &= \mathbb{I}\,\mathbb{M}\,\mathbb{I} \\ &= \textstyle\sum_{ij}\mathbb{P}_i\,\mathbb{M}\,\mathbb{P}_j \\ &= \textstyle\sum_{ij}|a_i)(A^i|\mathbb{M}|a_j)(A^j| \\ &= \textstyle\sum_{ij}m^i{}_j|a_i)(A^j| \quad \text{where} \quad m^i{}_j = (A^i|\mathbb{M}|a_j) \end{aligned} \tag{1}$$

Here $\mathbb{M}$ is displayed as a weighted linear combination of the n^2-population of matrices[2]

$$\mathbb{F}_{ij} = |a_i)(A^j| \quad : \quad \mathbb{F}_{ii} = \mathbb{P}_i$$

and $\|m^i{}_j\|$ provides the matrix representation with respect to the non-orthogonal $\{|a_i)\}$-basis of $\mathbb{M}$; it permits $|x) \to |\tilde{x}) = \mathbb{M}|x)$ to be represented

$$x^i \to \tilde{x}^i = m^i{}_j x^j$$

So much by way of preparation.

Let us assume—simply to keep simple things simple; both assumptions could easily be relaxed—that the eigenvalues (whence also the eigenvectors) of the otherwise arbitrary real square matrix $\mathbb{M}$ are real and distinct:

$$\mathbb{M}|e_i) = \lambda_i|e_i)$$

The eigenvectors $\{|e_i)\}$ then provide a generally non-orthogonal eigenbasis in $\mathcal{V}_n$. Proceeding as before, we construct $(e_i|e_j) = g_{ij}$, $(E^i| = g^{ij}|e_j)$ and the complete set of orthogonal projectors $\mathbb{P} = |e_i)(E^i|$. We then have, as an instance of (1),

$$\begin{aligned} \mathbb{M} &= \textstyle\sum_{ij}|e_i)(E^i|\mathbb{M}|e_j)(E^j| \\ &= \textstyle\sum_{ij}|e_i)\lambda_j\delta^i{}_j(E^j| \\ &= \textstyle\sum_i\lambda_i\mathbb{P}_i \end{aligned} \tag{2}$$

When $\mathbb{M}$ is symmetric the eigenbasis is orthogonal and (2) reduces to the familiar spectral decomposition, but as it stands it appears to provide an unrestricted generalization of that familiar result.

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[2] I have in the case $n = 2$ established the *trace-wise orthogonality* of the $\mathbb{F}$-matrices. I am confident that trace-wise orthogonality holds generally (proof would at the moment take me too far afield), which if so can be expected to support a rich "Fourier analytic" theory with diverse applications. It is my sense (see the Wikipedia article "Reciprocal Lattice") that Pontryagin and others—physicists as well as mathematicians—have already explored this ground, and pursued it to esoterically abstract heights.